ROTATION OF A CONDUCTING FLUID IN AN ANNULAR CLEARANCE UNDER THE INFLUENCE OF A TRANSVERSE MAGNETIC FIELD

(O VRASHCHENII PROVODIASHCHEI ZHIDKOSTI V KOL'TSEVOM ZAZORE PRI NALICHII POPERECHNOGO POLIA)

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Ia.S. UFLIAND
(Leningrad)

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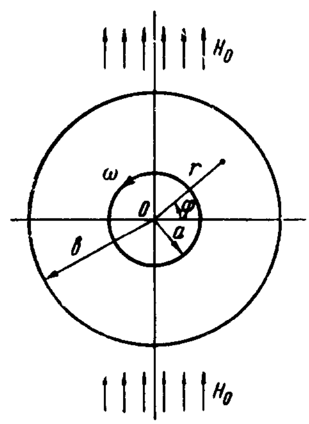
1. Statement of the problem. We deal with the steady motion of a viscous conducting incompressible fluid within the space bounded by two infinite cylinders of radii a and b ($a < b$).

It is assumed that the nonconducting internal cylinder rotates at constant angular velocity ω, the outside cylinder being stationary, whilst there is also an external homogeneous magnetic field $\mathbf{H}_0$ which acts perpendicular to the axes of the cylinders (Fig. 1).

Under the specified conditions, electric fields and currents act along the direction of the axis in the fluid, and ponderomotive forces act in the plane perpendicular to the axis. Additionally, the currents give rise to an induced magnetic field in the same plane. Thus the unknowns in the problem are the vector components of velocity $\mathbf{v}(r, \phi)$ and magnetic field $\mathbf{H}(r, \phi)$ in the plane perpendicular to the z-axis (r, ϕ, z, are cylindrical coordinates).



We note that it follows directly from the equation

$$\operatorname{rot} \mathbf{E} = 0 \qquad (1.1)$$

that $E = E_z = \text{const}$. It will be assumed that $E_0 = 0$ below, which allows us to lower the order of the magnetohydrodynamic equations [1] and bring them down to the form

$$\operatorname{rot}\mathbf{H} = \frac{4\pi\sigma}{c^2}\mathbf{v}\times\mathbf{H}, \qquad \operatorname{div}\mathbf{H} = 0, \qquad \operatorname{div}\mathbf{v} = 0 \tag{1.2}$$

$$\rho(\mathbf{v}\nabla)\mathbf{v} = \eta\triangle\mathbf{v} - \nabla p + \frac{1}{c}\mathbf{j}\times\mathbf{H}, \qquad \mathbf{j} = \frac{c}{4\pi}\operatorname{rot}\mathbf{H}$$

where σ is the conductivity of the fluid, η the viscosity coefficient, ρ the density, c the velocity of light, $\mathbf{j}$ the current-density vector, p the pressure.

Introduce the dimensionless variables

$$\mathbf{h} = \frac{\mathbf{H}}{H_0}, \qquad \mathbf{u} = \frac{\mathbf{v}}{v_0}, \qquad q = \frac{p}{p_0}, \qquad v_0 = \omega a, \qquad p = \rho v_c^2 \tag{1.3}$$
$$R = \frac{\rho}{\eta}v_0 a, \qquad R_m = \frac{4\pi\sigma}{c^2}v_0 a, \qquad M = \frac{H_0 a}{c}\sqrt{\frac{\sigma}{\eta}}$$

In these expressions, R is the Reynolds number, R_m is the magnetic Reynolds number, M is the Hartmann number. Making use of the relation

$$\mathbf{j}\times\mathbf{H} = \frac{\sigma}{c}[(\mathbf{v}\mathbf{H})\mathbf{H} - H^2\mathbf{v}]$$

Equation (1.2) can be transformed to

$$\begin{gathered}\operatorname{rot}\mathbf{h} = R_m\mathbf{u}\times\mathbf{h}, \qquad \operatorname{div}\mathbf{h} = 0, \qquad \operatorname{div}\mathbf{u} = 0\\ \triangle\mathbf{u} = R[(\mathbf{u}\nabla)\mathbf{u} + \nabla q] + M^2[h^2\mathbf{u} - (\mathbf{u}\mathbf{h})\mathbf{h}]\end{gathered} \tag{1.4}$$

Here the operations of differentiation are carried out with respect to the dimensionless variable $x = r/a$ which varies over the range $1 \leqslant x \leqslant \lambda = b/a$.

The system (1.4) contains four unknown functions $u_r(r, \phi)$, $u_\phi(r, \phi)$, $h_r(r, \phi)$, $h_\phi(r, \phi)$, and its solution requires eight boundary conditions.

Four of the conditions are self evident; they arise from the sticking of the fluid to the walls and are of the form

$$u_r|_{x=1} = u_r|_{x=\lambda} = u_\varphi|_{x=\lambda} = 0, \qquad u_\varphi|_{x=1} = 1 \tag{1.5}$$

The remaining boundary conditions should be obtained by solving the electrodynamic equations

$$\operatorname{rot}\mathbf{h} = \operatorname{div}\mathbf{h} = 0 \tag{1.6}$$

over the ranges $x < 1$ and $x > \lambda$, and the following conditions of continuity should be inserted:

$$h_r = h_r^{(a)}, \qquad h_\varphi = h_\varphi^{(a)} \quad \text{for } x = 1, \quad h_r = h_r^{(b)}, \qquad h_\varphi = h_\varphi^{(b)} \quad \text{for } x = \lambda \tag{1.7}$$

Here $h^{(a)}$ and $h^{(b)}$ are the magnetic-field strengths in the regions $r < a$ and $r > b$ respectively.

It is finally necessary to lay down the condition that the vector $h^{(a)}$ be bounded when $r \to 0$ whilst the vector $\mathbf{h}^{(b)}$ must have components $h_r^{(b)} = \sin\phi$, $h_\phi^{(b)} = \cos\phi$ for $r \to \infty$, corresponding to the given homogeneous field.

2. Approximate solution for small Reynolds and Hartmann numbers. We assume that the parameters R, R_m, M^2 are small quantities of similar magnitude. This occurs, for instance, in the case of a very viscous but weakly conducting fluid in a fairly strong field H_0. We write down

$$R_m = \varepsilon, \qquad R = \alpha\varepsilon, \qquad M^2 = \beta\varepsilon \tag{2.1}$$

where ϵ is a small quantity, α and β are finite quantities.

We look for the vectors $\mathbf{h}$ and $\mathbf{u}$, and also the quantity q in the form of an expansion

$$\mathbf{h} = \mathbf{h}_0 + \varepsilon\mathbf{h}_1 - \ldots, \qquad \mathbf{u} = \mathbf{u}_0 + \varepsilon\mathbf{u}_1 + \ldots, \qquad q = q_0 + \varepsilon q_1 + \ldots \tag{2.2}$$

It is obvious that the zero approximation for the magnetic field will be

$$h_{0r} \equiv \sin\varphi, \qquad h_{0\varphi} = \cos\varphi \tag{2.3}$$

over the whole space. Furthermore, from the equation

$$(\triangle_0\mathbf{u})_r = \triangle u_{0r} - \frac{u_{0r}}{x^2} - \frac{2}{x^2}\frac{\partial u_{0\varphi}}{\partial\varphi} = 0, \quad (\triangle\mathbf{u}_0)_\varphi = \triangle u_{0\varphi} - \frac{u_{0\varphi}}{x^2} + \frac{2}{x^2}\frac{\partial u_{0r}}{\partial\varphi} = 0 \tag{2.4}$$

the zero approximation for the velocity emerges:

$$u_{0r} \equiv 0, \qquad u_{0\varphi} = \frac{\lambda^2 - x^2}{x(\lambda^2 - 1)} \tag{2.5}$$

which satisfies all the boundary conditions and corresponds to a purely hydrodynamic regime. We will now turn to finding the next approximation.

In order to estimate the influence of the current on the magnetic field in the first approximation, it is necessary to solve the equations

$$\operatorname{rot}\mathbf{h}_1 = \mathbf{u}_0 \times \mathbf{h}_0, \qquad \operatorname{div}\mathbf{h}_1 = 0 \tag{2.6}$$

or, in terms of components

$$\frac{\partial}{\partial x}(xh_{1r}) + \frac{\partial h_{1\varphi}}{\partial \varphi} = 0, \qquad \frac{1}{x}\frac{\partial}{\partial x}(xh_{1\varphi}) - \frac{1}{x}\frac{\partial h_{1r}}{\partial \varphi} = \frac{x^2 - \lambda^2}{x(\lambda^2 - 1)}\sin\varphi \tag{2.7}$$

Assuming

$$h_{1r} = A(x)\frac{\cos\varphi}{1-\lambda^2}, \qquad h_{1\varphi} = B(x)\frac{\sin\varphi}{1-\lambda^2} \tag{2.8}$$

from (2.7) we find

$$A(x) = \frac{x^2}{8} - \frac{\lambda^2}{2}\ln x + \frac{C}{x^2} + D, \qquad B(x) = -\frac{3x^2}{8} + \frac{\lambda^2}{2}\ln x + \frac{C}{x^2} - D \tag{2.9}$$

To determine the constants C and D we use the solution of the homogeneous system (2.7) over the ranges $x < 1$ and $x > \lambda$

$$h_{1r}^{(a)} = D^{(a)}\frac{\cos\varphi}{1-\lambda^2}, \quad h_{1\varphi}^{(a)} = -D^{(a)}\frac{\sin\varphi}{1-\lambda^2}, \qquad h_{1r}^{(b)} = \frac{C^{(b)}}{x^2}\frac{\cos\varphi}{1-\lambda^2}, \qquad h_{1\varphi}^{(b)} = \frac{C^{(b)}}{x^2}\frac{\sin\varphi}{1-\lambda^2} \tag{2.10}$$

and we find C, D, $C^{(b)}$ and $D^{(a)}$ from the conjugate conditions (1.7).

We find on calculation that

$$A(x) = \frac{1}{8}\left[x^2 + \frac{1-2\lambda^2}{x^2}\right] + \frac{\lambda^2}{2}\ln\frac{\lambda}{x}, \quad B(x) = -\frac{1}{8}\left[3x^2 + \frac{2\lambda^2-1}{x^2}\right] + \frac{\lambda^2}{2}\left(1 - \ln\frac{\lambda}{x}\right) \tag{2.11}$$

which represents the first approximation for the magnetic field.

We now turn to the determination of the effect of the magnetic field on the motion of the fluid to a first approximation.

The functions $u_{1r}(x, \phi)$, $u_{1\phi}(x, \phi)$ and $q_0(x, \phi)$ must be found from the following system of differential equations:

$$\begin{gathered}\Delta u_{1r} - \frac{u_{1r}}{x^3} - \frac{2}{x^2}\frac{\partial u_{1\varphi}}{\partial\varphi} = -\alpha\left(\frac{u_{0\varphi}^2}{x} - \frac{\partial q_0}{\partial x}\right) - \frac{\beta}{2}u_{0\varphi}\sin 2\varphi \\ \Delta u_{1\varphi} - \frac{u_{1\varphi}}{x^2} + \frac{2}{x^2}\frac{\partial u_{1r}}{\partial\varphi} = \frac{\alpha}{x}\frac{\partial q_0}{\partial\varphi} + \frac{\beta}{2}u_{0\varphi}(1-\cos 2\varphi) \quad \frac{\partial}{\partial x}(xu_{1r}) + \frac{\partial u_{1\varphi}}{\partial\varphi} = 0\end{gathered} \tag{2.12}$$

with the boundary conditions

$$u_{1r} = u_{1\varphi} = 0 \qquad \text{for} \quad x = 1 \text{ and } x = \lambda \tag{2.13}$$

We look for a solution to system (2.12) in the form

$$u_{1r} = \beta R(x, \varphi), \quad u_{1\varphi} = \beta\psi(x) + \beta\Phi(x, \varphi), \quad q_0 = \chi(x) + \frac{\beta}{\alpha}f(x, \varphi) \tag{2.14}$$

The functions $\psi(x)$, $R(x, \phi)$, $\Phi(x, \phi)$ and $f(x, \phi)$ satisfy the equations

$$\frac{1}{x}(x\psi')' - \frac{\psi}{x^2} = \frac{u_{0\varphi}}{2} \tag{2.15}$$

$$\triangle R - \frac{R}{x^2} - \frac{2}{x^2}\frac{\partial\Phi}{\partial\varphi} = \frac{\partial f}{\partial x} - \frac{u_{0\varphi}}{2}\sin 2\varphi, \qquad \triangle\Phi - \frac{\Phi}{x^2} + \frac{2}{x^2}\frac{\partial R}{\partial\varphi} = \frac{1}{x}\frac{\partial f}{\partial\varphi} - \frac{u_{0\varphi}}{2}\cos 2\varphi, \qquad \frac{\partial}{\partial x}(xR) + \frac{\partial\Phi}{\partial\varphi} = 0 \tag{2.16}$$

The functions $\psi(x)$, R and Φ should vanish when $x = 1$ and $\kappa = \lambda$ whilst the function $\chi(x)$ is determined by the formula

$$\chi'(x) = \frac{u_{0\varphi}^2}{x} \tag{2.17}$$

and this evidently represents the part of the pressure which balances the centrifugal forces which arise due to the velocity $u_{0\phi}$.

The function $\psi(x)$ can be found easily and has the form

$$\psi(x) = \frac{1}{16x(\lambda^2-1)^2}\{4\lambda^2[(\lambda^2-1)x^2\ln x - (x^2-1)\lambda^2\ln\lambda] + (\lambda^2-1)(\lambda^2-x^2)(x^2-1)\} \tag{2.18}$$

If we differentiate the first of Equations (2.16) with respect to ϕ, we multiply the second equation of (2.16) by x and differentiate with respect to x, subtract the second result from the first and combine the result with the last equation (2.16); we arrive at a system of two equations with unknown functions R and ϕ. If we put

$$R = y(x)\sin 2\varphi, \qquad \Phi = Z(x)\cos 2\varphi \tag{2.19}$$

we separate the particular solution of the system of ordinary differential equations so obtained and eliminate Z, then for the function $t = xy$ we obtain the equation

$$x^3 t^{\mathrm{IV}} + 2x^2 t''' - 9xt'' + 9t' = 0 \tag{2.20}$$

with the general integral

$$t = Ax^2 + Bx^4 + \frac{C}{x^2} + D \tag{2.21}$$

On determining the constants from the boundary conditions we arrive at the following formulas after some calculation:

$$y(x) = \frac{\lambda^2}{16x^3(\lambda^2-1)^4}\{2\lambda^2(x^2-1)^2[\lambda^2(\lambda^2+1)-2x^2]\ln\lambda - 2x^4(\lambda^2-1)^3\ln x - (\lambda^2-1)(x^2-1)(\lambda^2-x^2)[x^2(\lambda^2+1)-2\lambda^2]\} \tag{2.22}$$

$$Z(x)=\frac{\lambda^2}{8x^3(\lambda^2-1)^4}\{\lambda^2(x^2-1)[\lambda^2(\lambda^2+1)(x^2+1)-4x^4]\ln\lambda-$$
$$-x^4(\lambda^2-1)^3\ln x-(\lambda^2-1)(x^2-1)(\lambda^2-x^2)[\lambda^2(x^2+1)+x^2]\} \qquad (2.23)$$

There only remains now the determination from Equation (2.16) of the components $\partial f/\partial x$ and $x^{-1}\ \partial f/\partial\phi$ of the pressure gradient, and the solution to the first approximation is finished. (Here only the zero approximation is obtained for the pressure.)

The next approximation can be sought in a similar way, but the actual computation turns out to be rather complicated even for the second approximation.

3. Calculation of the moment. We make use of the results obtained in order to find the turning moment which must be applied to the cylinder in order to balance the forces of viscous friction and establish a uniform rotation.

If we make use of the expression for the frictional stress F_a on the surface of the rotating cylinder

$$F_a=-\eta\left(\frac{\partial v_\varphi}{\partial r}-\frac{v_\varphi}{r}+\frac{1}{r}\frac{\partial v_r}{\partial\varphi}\right)_{r=a} \qquad (3.1)$$

we find

$$\frac{F_a}{F_a^{(0)}}=1-\frac{M^2}{2\lambda^2}(\lambda^2-1)[\psi'(1)+Z'(1)\cos 2\varphi] \qquad (3.2)$$

In this expression

$$F_a^{(0)}=\frac{\eta v_0}{a}\frac{2\lambda^2}{\lambda^2-1},\qquad \psi'(1)=\frac{(3\lambda^2-1)(\lambda^2-1)-4\lambda^2\ln\lambda}{8(\lambda^2-1)^2} \qquad (3.3)$$

$$Z'(1)=\lambda^2\frac{4\lambda^2(\lambda^2+2)\ln\lambda-(\lambda^2-1)(5\lambda^2+1)}{8(\lambda^2-1)^3}$$

whilst $F_a^{(0)}$ is the frictional stress at $r=a$ in the corresponding ordinary hydrodynamic problem.

It can be shown that for all values of $\lambda>1$ the quantity $\psi'(1)+Z'(1)<0$ whilst $Z'(1)>0$. Thus magnetohydrodynamic effects in this case always increase the friction on the rotating surface, as would be expected, because in this problem there is no electric field.

If we use (3.2) we obtain a formula for the moment

$$\frac{L_a}{L_a^{(0)}}=1+M^2 f_a(\lambda),\qquad L_a^{(0)}=4\pi\eta v_0 a\frac{\lambda^2}{\lambda^2-1} \qquad (3.4)$$

and the function

$$f_a(\lambda) = \frac{4\lambda^4 \ln \lambda - (3\lambda^2 - 1)(\lambda^2 - 1)}{16\lambda^2(\lambda^2 - 1)} \qquad (3.5)$$

remains positive for all values of $\lambda > 1$.

We also deduce formulas for the stress and turning moment on the stationary surface $r = b$

$$\frac{F_b}{F_b^{(0)}} = 1 - \frac{M^2}{2}(\lambda^2 - 1)[\psi'(\lambda) + z'(\lambda)\cos 2\varphi], \qquad F_b^{(0)} = -\frac{\eta v_0}{a}\frac{2}{\lambda^2 - 1}$$
$$\frac{L_b}{L_b^{(0)}} = 1 - M^2 f_b(\lambda), \qquad f_b(\lambda) = \frac{\lambda^4 - 1 - 4\lambda^2 \ln \lambda}{16(\lambda^2 - 1)}, \qquad L_b^{(0)} = -L_a^{(0)} \qquad (3.6)$$

Here the quantities $z'(\lambda)$, $\psi'(\lambda) - z'(\lambda)$ and $f_b(\lambda)$ will be positive for all values of $\lambda > 1$. This means that on the stationary surface, the magnetic field has the effect of reducing friction.

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